A Construction of the Principal Period-2 Component in the Degree-9 Bifurcation Set with Parametric Boundaries

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9차 분기집합의 2-주기 성분의 경계방정식에 관한 연구

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Abstract By extending the Mandelbrot set for the complex polynomial
\[ M = \left\{ c \in \mathbb{C} : \lim_{k \to \infty} P_c^k(0) = \infty \right\} \]
we define the degree-n bifurcation set. In this paper, we formulate the boundary equation of a period-2 component on the main component in the degree-9 bifurcation set by parameterizing its image. We establish an algorithm constructing a period-2 component in the degree-9 bifurcation set and the typical implementations show the satisfactory result with Mathematica codes grounded on the analysis.

Key words : degree-n bifurcation set, principal period-2 component, ray of symmetry

요약 본 논문은 맨델브로트 집합을 9차 복소 다항식에 확장시켜 새로운 프랙탈 도형을 나타내는 9차 분기집합을 정의하고, 2주기 성분의 경계방정식을 해석함으로써 표 현한다. 또한, 2주기 성분을 적도하는 알고리즘을 고안하고, 메스 메티를 활용하여 2주기 성분의 기하학적 구조에 관한 결과를 제시하고자 한다.

1. Introduction

The Mandelbrot set[1,2] is defined to be the set of all complex values of \( c \) such that the critical orbit under the complex polynomial \( P_c(z) = z^2 + c \) does not escape to infinity, where \( c \in \mathbb{C} \) denotes the sets of complex numbers. The notion of the Mandelbrot set was generalized to a complex polynomial of degree \( n \geq 2 \) and this generalized Mandelbrot set is called the degree-n bifurcation set introduced by Devaney[1]. We investigate the boundary equation of \( M \) and the boundary of the principal period-2 bulb in this proposed set.

As beginning studies, we will describe some preliminary definitions and theorems concerning the geometric properties of the degree-n bifurcation set. Let \( R \) and \( N \) denote the sets of real numbers and natural numbers, respectively.

Definition 1.1 Let \( P_c(z) = z^n + c \) for an integer \( n \geq 2 \) with \( c, z \in \mathbb{C} \). Then the degree-\( n \) bifurcation set is defined to be the set
\[ M = \left\{ c \in \mathbb{C} : \lim_{k \to \infty} P_c^k(0) = \infty \right\} \]

where \( P_{k+1}(z) = P(P_k(z)) \) is the \( k \)-fold composite map of \( P \) at \( z \) with \( P^0(z) = z \). This definition was introduced by Devaney[1] in 1986.

Definition 1.2 The sets
\[ P_m = \left\{ c \in \mathbb{C} : r e^{i \phi_m}, r \geq 0, \phi_m = m \pi / (n-1) \right\} \quad \text{for} \quad (m = 1, 2, \ldots, 2n-2) \]
called the rays of symmetry and \( P_1 \) is called the principal ray of symmetry and
denoted by $P$. The set
\[ S = \{ c \in C : c = re^{i\theta}, r \geq 0, 0 < \theta \leq \pi/(n-1) \} \]
is called the principal sector.

**Definition 1.3** An attracting period-$k$ component[3]

is defined by the following set
\[ \{ c \in C : \text{there exist } \xi \text{ such that } P^k_{c(\xi)} - c \left| \frac{d}{dz} P^k_{c(\xi)} \right| < 1 \} \]
and is denoted by $M_k$. Because the attracting period-2 component $M_k$ lies on every odd ray of symmetry, the $M_k$ lying on the principal ray of symmetry is called the principal period-2 component.

The following theorems convinces the symmetry of the degree-$n$ bifurcation set with respect to rays of symmetry in the complex plane.

**Theorems 1.1** The degree-$n$ bifurcation set $M$ is symmetric about rays $P_m$ for all $m = 1, 2, \cdots, 2n-2$ in the $c$-parameter plane.

Proof. See p. 224, Geum and Kim[4,5].

2. Formulating the Boundary of a Period-2 Component

In this section, we will describe the boundary equation of a period-2 component on the main component in the degree-9 bifurcation set by parameterization. Let $P_c(w) = w^9 + c$ with $z, c, w \in C$. The boundary equation of this propose component is as follows:

\[ P_c^2(z) = z \]

where $\lambda = \left| \frac{d}{dw} P_c^2(w) \right|_{w=z}$ is a multiplier at $z$, which is a period-2 point of $P_c$. Since (1) describes a circle in the image plane, we use a parameter $\phi \in [0, 2\pi)$. From the complex analysis, (2) is represented by

\[ z(\psi^0 + c) = \left( \frac{1}{g^2} \right)^{1/8} e^{i\phi_j}, \quad \text{for } j \in \{0, 1, 2, \cdots, 7\}. \]

where $\psi_j = \frac{\phi_j + 2j\pi}{8}$. Let $a = \left( \frac{1}{g^2} \right)^{1/8} e^{i\phi_j}$, for $j \in \{0, 1, 2, \cdots, 7\}$. Then we obtain the following equation for $z(\psi^0 + c) = a$.

Combining (1) with (4), we get

\[ c = z - (a/z)^9 = a/z - z^9 \quad \text{and} \quad z^9 - (a/z)^9 + z - a/z = 0. \]

\[ \left\{ z - \frac{a}{z}, z^2 + \frac{a}{z^2} + \frac{a^2}{z^3}, \cdots, \frac{a^8}{z^8} + \frac{a^9}{z^9}, \frac{1}{z} \right\} = 0 \]

Since $z - a/z = 0$ means a fixed point, we choose the period-2 point of $P_c$ satisfying the following:

\[ z^8 + z^7 \left( \frac{a}{z} + \frac{a^2}{z^2} \right) + \cdots + \frac{a^8}{z^8} + 1 = 0 \]

**Regrouping** (5) yields

\[ z^9 + a(z^8 + \frac{a}{z^8}) + a^2(z^7 + \frac{a^2}{z^7}) + \cdots + a^8(z + \frac{a^8}{z}) + a^9 + 1 = 0 \]

From the point $c$ on the boundary satisfies $c = a/z - z^9 = a/z - z(\psi^0)$, we have as follows:

\[ c = a(z^7 + \frac{a}{z^7}) + a^2(z^6 + \frac{a^2}{z^6}) + \cdots + a^8(z + \frac{a^8}{z}) + a^9 + \frac{a^9}{z} \]

By Vieta's transformation $t = z + a/z$, we express Eq.(6) simply. We define the function $F_k$ for $k \geq 0$ by means of the equation

\[ F_k = z^k + \frac{a^k}{z^k}, \quad F_0 = 0, \quad F_1 = z + \frac{a}{z} \]

It is evident from Eq(7) that

\[ (z^k + \frac{a^{k-1}}{z^k})(z + \frac{a}{z}) = z^k + \frac{a^k}{z^k} + a(z^{k-2} + \frac{a^{k-2}}{z^k}) \]

That is,

\[ F_k = t \cdot F_{k-1} - a \cdot F_{k-2}, \quad \text{for } k \geq 2. \]

Evidently, we obtain for $k \geq 2$,

\[ F_k(t, a) = t^k - a \sum_{j=0}^{k-2} t^{k-2-j} \cdot F_j. \]

We shall derive the following proposition by the induction $m \geq 1$.

**Proposition 1.1** Let $s = t^2 - 2a$ for fixed $a$ and $s, t, a \in C$. Let $G_m(s)$ and $H_m(s)$ be polynomials in $s$ of degree $m$ and defined by means of the
equations
\[
\begin{align*}
G_{m+1}(s) &= (s + 2\alpha)H_m(s) - aG_m(s) \\
H_{m+1}(s) &= G_{m+1}(s) - aH_m(s)
\end{align*}
\]
where
\[
G_1 = F_2 = s, \quad H_0 = 1, \quad \text{and} \quad H_i = H_i - aH_0 = s - a.
\]
Then from definition \(F_m\), it follows that, for \(m \geq 1\)
\[
\begin{align*}
F_{2m} &= G_m(s), \\
F_{2m+1} &= t \cdot H_m(s)
\end{align*}
\]  \(11\)

The above properties lead us to simplify the system of equations.

According to the equation \(11\), we can write Eq \(6\) as
\[
e = t \left( 1 + \sum_{\nu=1}^{k} a^\nu \cdot H_{k-\nu}(s) \right)
\]  \(12\)

We find that \(c(t, (a))\) and \(c(s, (a))\) as a function of \(a(\psi)\) is each branch of \(c\) in the statement of Eq \(12\). The symmetry of the degree-9 bifurcation set \(M\) with respect to the rays \(m\pi/8\), for \(m = 1, 2, \cdots, 16\) and the continuity of the boundary enable us to consider 16 circular arcs lying on the ray \(m\pi/8\).

Let \(n \in \mathbb{N} - \{1\}\). Now the circular arcs are defined by writing
\[\Gamma_0 = \left\{ a \in \mathbb{C} : a = \left( \frac{1}{\sqrt{2}} \right)^n e^{i\theta}, \frac{j\pi}{4} \leq \theta < \frac{(j+1)\pi}{4} \right\} \text{ for } j = 0, 1, \cdots, 7,\]
and \(c(t, (a))\) has 4 circular arcs.

We are interested in the period-2 component lying on the principal ray of symmetry. It is enough to choose \(a \in \Gamma_0\) to consider the half of the boundary of the period-2 component.

3. Algorithm and Results

We establish an algorithm drawing the boundaries of period-2 components in this proposed set with Mathematica[6].

Algorithm 3.1
Step 1. Find the respective solution \(t, (a), s, (a)\) of Eq. (5).

Step 2. Evaluate each branch \(c_i\), for \(i = 1, 2, \cdots, 4\) of \(c\) based on \(t, (a)\) found in Step 1.

Step 3. Put \(x_i = \Re(c_i(a(\psi)))\) and \(y_i = \Im(c_i(a(\psi)))\) for \(i = 1, 2, \cdots, 4\). Plot the curve \(c_i\) as the set
\[\left\{ (x_i(a(\psi)), y_i(a(\psi))) : 0 \leq \psi \leq \frac{\pi}{4}, 1 \leq i \leq 4 \right\}\]

Step 4. Rotate \(c_i\) obtained in Step 3 by an angle \(j\pi/4\) for \(j \in \{0, 1, 2, 3\}\). The rotated curves denoted by \(G_i\) is a part of the boundary of the principal period-2 component.

Step 5. Construct \(G_2\), the symmetric curves of \(G_i\).

Step 6. Complete the boundary curve \(G_1 \cup G_2\).

The statements in Section 2 are used to draw the principal period-2 component. Typical boundary and branches of \(M^c\) are shown in Figure 1 with \(n = 9\).
The current analysis shown in this paper can be departmentalized to parameterize the boundary equation of the period-3 component in the degree-9 bifurcation.

References


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<Major Research Area>
Fractal sets, Numerical Analysis, Degree-n Bifurcation set, root-finding algorithm,.....